(c) Show that

Q. Maxwell equation is invariant under Lorentz transformation.

We have the set of Maxwell eqn is

$$\nabla \cdot E = 0 \quad \text{(i)}$$

$$\nabla \cdot B = 0 \quad \text{(ii)}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \quad \text{(iii)}$$

$$\nabla \times B = \mu_0 \sigma + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \quad \text{(iv)}$$

The equation (ii) and (iii) do not contain the quantities ε_0, μ_0, ρ or J and have the same form in vacuum as in a material medium. They contain the field quantities E and B.

The spatial component of vector quantities of eqn (2) and (3) can be written as.

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \quad \text{(iia)}$$

&

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \quad \text{(iiia)}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t} \quad \text{(iiib)}$$

&

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t} \quad \text{(iiic)}$$

These equations are a set of coupled partial differential equation containing the component of E and B field & apply at a point (x, y, z) at time t in the inertial frame S.

We are to prove that they have the same form in inertial frame S'. Where S' frame is moving with uniform translational velocity v with respect to S. along the common $x - x'$ axis.

② → invariance of Maxwell equⁿ

(2)

Now let the magnetic field and electric field is denoted by B' and E' in the frame S' and they satisfy the Maxwell's 2nd and 3rd equation as

$$\left.\begin{aligned} \nabla \cdot B' &= 0 \\ \nabla \times E &= -\partial B'/\partial t' \end{aligned}\right\} \longrightarrow (V)$$

In component form the two equations are

$$\frac{\partial B_x'}{\partial x'} + \frac{\partial B_y'}{\partial y'} + \frac{\partial B_z'}{\partial z'} = 0 \longrightarrow (VI)$$

$$\frac{\partial E_z'}{\partial y'} - \frac{\partial E_y'}{\partial z'} = -\frac{\partial B_x'}{\partial t'} \longrightarrow (VII)$$

$$\frac{\partial E_x'}{\partial z'} - \frac{\partial E_z'}{\partial x'} = -\frac{\partial B_y'}{\partial t} \longrightarrow (VIII)$$

$$\& \quad \frac{\partial E_y'}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t} \quad (IX)$$

We have to know the relation between the partial differentiation with respect to one set of variables to the corresponding partial differentiation w.r.to the other set of variables (x', y', z', t') The variables are connected by the Lorentz transformation equation

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

Now the relation between the diffent components of primed and unprimed frame is as follows

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial}{\partial y'} + \frac{\partial z'}{\partial x}\cdot\frac{\partial}{\partial z'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'}$$

Now $\frac{\partial x'}{\partial x} = \gamma, \quad \frac{\partial y'}{\partial x} = 0, \quad \frac{\partial z'}{\partial x} = 0, \quad \frac{\partial t'}{\partial x} = -\frac{\gamma v}{c^2}$

③ invariants of Maxwell equations

③/P

$$\frac{\partial}{\partial x} = \gamma\left(\frac{\partial}{\partial x'} - \frac{v}{c^2}\frac{\partial}{\partial t'}\right)$$

$$\text{Ly}\ \frac{\partial}{\partial y} = \frac{\partial}{\partial y'}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z'}$$

$$\& \quad \frac{\partial}{\partial t} = \gamma\left(\frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'}\right)$$

} → (x)

Substituting the equ^{n} (x) in (iia) & (iiid) → (iii c) we have

$$\frac{\partial}{\partial z'}E_x - \frac{\partial}{\partial x'}\left[\gamma(E_z + vB_y)\right] = -\frac{\partial}{\partial t'}\left[\gamma\left(B_y + \frac{vE_z}{c^2}\right)\right] \rightarrow (xi)$$

Now the equation (iii a) → (iii d) can be written

$$\frac{\partial E_x'}{\partial x} - \frac{\partial}{\partial x'}E_z = \frac{\partial}{\partial t'}B_y'$$

This relation is only possible ~~to~~ if we put

$$E_x' = E_x \qquad E_y' = \gamma(E_z + vB_y) \qquad B_y' = \gamma\left(B_y + \frac{vE_z}{c^2}\right)$$

~~& $B_y' = \gamma(B_y + vE_z$~~

→ (xi)

These relations ~~show~~ are already the ~~transformation~~ transformation of E & B field.

In the same way, & if we substitute the partial relation (x), we have

$$E_x' = E_x$$

$$E_y' = \gamma(E_y - vB_x)$$

$$B_z' = \gamma\left(B_z - \frac{vE_y}{c^2}\right)$$

Now we find how B_x transforms.

Comparing the equ^{n} (vii) – (x) and (iia) to (iiid) we have, we see that the two equation

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To interchange if we ~~place~~ Set $B_x' = B_x$ which completes the set of relation of transformation of the field components.

∴ When we transform the space time Co-ordinates by means of Lorentz transformation equations, we find that Maxwell equations are invariant in form.

Relativistic transformation of $\vec{E}$ and $\vec{B}$

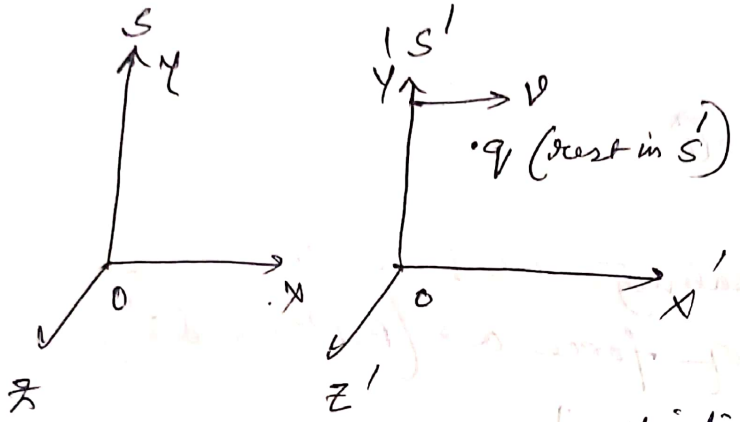
We have the electromagnetic force on a particle moving with velocity v and charge q is

$$F = q(E + v \times B) \longrightarrow (1)$$

I. Electric force on a charge q having field intensity E is

$$F_e = qE$$

II. Magnetic force on a charge q moving with velocity v in a magnetic field B is $F_B = q(v \times B)$



Description of the situation

1. S and S' are two inertial frame where S' is moving with const translational velocity v w.r. to S along +ve x direction.

2. q a charge is at rest w.r. to S'.
∴ The electric field in S' is E' and magnetic field B'.

$F_e' = qE'$

$F_B' = 0$ as charge is at rest

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3. For the same charge w.r.to S

The charge is now in motion with constant translational velocity v along x x' axes.

∴ The Three component of velocity of the charge particle is

$$\left.\begin{aligned} v_x &= v \\ v_y &= 0 \\ v_z &= 0 \end{aligned}\right\} \longrightarrow (1)$$

According to relativity we have the transformation of force as (Between S and S')

$$\left.\begin{aligned} F_x' &= F_x \\ F_y' &= \frac{F_y}{\sqrt{1 - v^2/c^2}} = \gamma F_y \\ F_z' &= \gamma F_z \end{aligned}\right\} (2)$$

Now for the present case

$$F_x = q\left[E_x + (v \times B)_x\right]$$

$$= q\left[E_x + \underbrace{(v_y B_z - v_z B_y)}_{\downarrow\downarrow \; 0}\right]$$

$$= qE_x$$

$$\therefore F_x = F_x' = qE_x'$$

$$\Rightarrow E_x = E_x' \longrightarrow (3)$$

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$F_y = qE_y$

Using the transformation relation

$$F_y = \frac{F_y'}{\gamma} = \frac{qE_y'}{\gamma}$$

$$\Rightarrow \frac{qE_y'}{\gamma} = q\left(E_y + (v \times B)_y\right)$$

$$\Rightarrow E_y' = \gamma\left(E_y + (v_x B_z - v_z B_x)\right)$$

($v_z = 0$)

$$\Rightarrow E_y' = \gamma\left(E_y - vB_z\right) \longrightarrow (4)$$

$$\text{By } E_z' = \gamma\left(E_z + vB_y\right) \longrightarrow (5)$$

Combining (3), (4) and (5)

$$E_x' = E_x$$
$$E_y' = \gamma\left(E_y - vB_z\right)$$
$$E_z' = \gamma\left(E_z + vB_y\right)$$

$$E_x = E_x'$$
$$E_y = \gamma\left(E_y' + vB_z'\right)$$
$$E_z = \gamma\left(E_z' - vB_y'\right)$$

$$E_{\parallel}' = E_{\parallel}$$
$$E_{\perp}' = \gamma\left(E_{\perp} + (v \times B)_{\perp}\right) \longrightarrow (6)$$

The magnetic component of force follows the following set of transformation.

$$B_x' = B_x$$
$$B_y' = \gamma\left(B_y + \frac{v}{c^2} E_z\right)$$
$$B_z' = \gamma\left(B_z - \frac{v}{c^2} E_y\right)$$

$$B_x'' = B_x'$$
$$B_y'' = \gamma\left(B_y' - \frac{v}{c^2} B_z'\right)$$
$$B_z'' = \gamma\left(B_z' - \frac{v}{c^2} E_y'\right)$$

Combining the above three equations

$$B_{\parallel}' = B_{\parallel}$$
$$B_{\perp}' = \gamma\left[B_{\perp} - \frac{1}{c^2}(v \times E)_{\perp}\right] \quad \} (7)$$